Synchronization in Power Networks and in Non-uniform Kuramoto Oscillators

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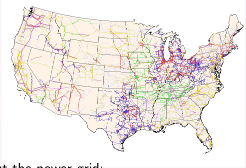
Intro: Transient Stability in Power Networks

"The vast North American power grid is the largest and most complex machine engineered by humankind." [P. Kundur '94, V. Vittal '03, ...]



Quick facts about the power grid:

- large-scale, complex, and nonlinear

⇒ various dynamic phenomena and instabilities

- 100 years old and operating at its capacity limits

⇒ increasing number of blackouts: New England '03, Italy '03, Brazil '09

Intro: Transient Stability in Power Networks

The New York Times

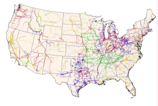
THE BLACKOUT OF 2003: Failure Reveals Creaky System, Experts Believe 8/15/2003

Energy is one of the top three national priorities [B. Obama, '09]

Expected additional synergetic effects in future "**smart grid**":

⇒ increasing complexity and renewable stochastic power sources

⇒ increasingly many transient disturbances to be detected and rejected



Transient Stability: Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of

- transmission lines and components,
- generation or load.

Intro: Transient Stability Analysis in Power Networks

Mathematical model of a power network:

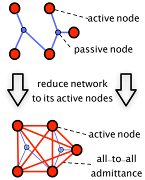
- swing equation for generator i:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{mi} - P_{ei}$$

$\theta(t)$ is measured w.r.t. a 60Hz rotating frame

- network-preserving model leads to DAEs
- network-reduction model leads to ODEs with reduced admittance matrix $Y_{ij} = |Y_{ij}|\, e^{i(\frac{\pi}{2}-\varphi_{ij})}$

$$P_{ei} = E_i^2 G_{ii} + \sum\nolimits_{j\neq i} E_i E_j\, |Y_{ij}| \sin(\theta_i - \theta_j + \varphi_{ij})$$



Classic model considered in transient stability analysis:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

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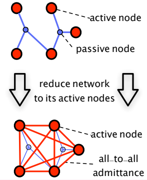

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Transient stability and synchronization:

- frequency equilibrium: $(\dot{\theta}_i, \ddot{\theta}_i) = (0,0)$ for all i
- synchronous equilibrium: $\theta_i - \theta_j$ bounded & $\dot{\theta}_i - \dot{\theta}_j = 0$ for all $\{i,j\}$

Classic problem setup in transient stability analysis:

1. power network in stable frequency equilibrium
2. $\rightarrow$ transient network disturbance and fault clearance
3. stability analysis of a new frequency equilibrium in post-fault network

More general synchronization problem:

synchronization in presence of transient network disturbances

Classic model considered in transient stability analysis:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Transient stability and synchronization:

synchronization in presence of transient network disturbances

Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla_i U(\theta)^T$$

Energy function analysis, (extended) invariance principle, analysis of reduced gradient flow [N. Kakimoto et al. 78, H.-D. Chiang et al. '94]

$$\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Key objective: compute domain of attraction via numerical methods

Classic model considered in transient stability analysis:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Transient stability and synchronization:

synchronization in presence of transient network disturbances

Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla H_i(\theta)^T \quad \rightsquigarrow \quad \dot{\theta}_i = -\nabla_i U(\theta)^T$$

$\Rightarrow$ **Open problem** [D. Hill and G. Chen '06]: power sys $\overset{?}{\leftrightsquigarrow}$ network:

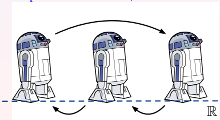
transient stability, performance, and robustness of a power network $\overset{?}{\leftrightsquigarrow}$ underlying network topology, parameters, and state

Detout - Consensus Protocols & Kuramoto Oscillators

Consensus protocol in $\mathbb{R}^n$:

$$\dot{x}_i = -\sum_{j \neq i} a_{ij}(x_i - x_j)$$

- n identical **agents** with state variable $x_i \in \mathbb{R}$
- **graph** with globally reachable node and weights $a_{ij} > 0$
- **objective** is state agreement: $x_i(t) - x_j(t) \to 0$
- **application:** social networks, computer science, systems theory
  robotic rendezvous, distributed computing, filtering and control ...
- **some references:** [M. DeGroot '74, J. Tsitsiklis '84, ...]

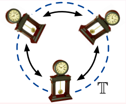


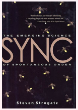
Detour - Consensus Protocols & Kuramoto Oscillators

Kuramoto model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

- **oscillators** with phase $\theta_i \in \mathbb{T}$, frequency $\omega_i \in \mathbb{R}$, **complete** coupling
- **objective** is synchronization: $\theta_i(t) - \theta_j(t)$ bounded, $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
- **application:** physics, biology, engineering
  coupled neurons, Josephson junctions, motion coordination ...
- **some references:** [Y. Kuramoto '75, A. Winfree '80, ...]





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$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

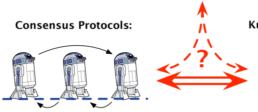
- degrees of synchronization:
  1. frequency entrainment
  2. phase locking
  3. phase synchronization
- known that
  1. for large K, frequency entrainment & phase locking
  2. additionally, for $\omega_i = \omega_j$, phase synchronization

Intro: The Big Picture

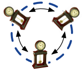
Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Consensus Protocols: **Kuramoto Oscillators:**



$$\dot{x}_i = -\sum_{j \neq i} a_{ij}(x_i - x_j) \qquad \dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j \neq i}\sin(\theta_i - \theta_j)$$



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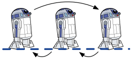
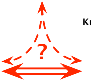
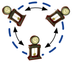
Consensus Protocols:





Kuramoto Oscillators:



$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i-x_j) \qquad \dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i-\theta_j)$$

Possible connection has often been hinted at in the literature!

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97]

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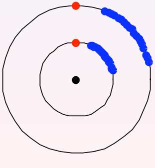
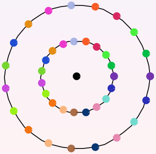
   1. synchronization and transient stability
   2. power network model
   3. consensus and Kuramoto oscillators
2. Singular perturbation analysis
   (to relate power network and Kuramoto model)
3. Synchronization analysis (of non-uniform Kuramoto model)
   1. Main synchronization result
   2. Sufficient condition (based on weakest lossless coupling)
   3. Sufficient condition (based on lossless algebraic connectivity)
   4. Further results
4. Conclusions

From the swing equations to the Kuramoto model

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i-\theta_j+\varphi_{ij})$$

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i-\theta_j) \quad \Longrightarrow \quad D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i-\theta_j+\varphi_{ij})$$

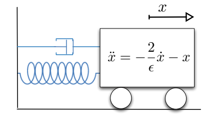


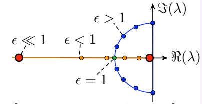


Singular Perturbation Analysis

Time-scale separation in power network model:

- **Motivation:** harmonic oscillator





for $\epsilon \ll 1 \Rightarrow$ two time-scales

- **Singular perturbation analysis:**

Singular Perturbation Analysis

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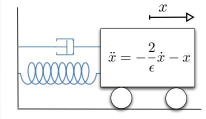

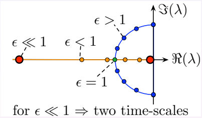

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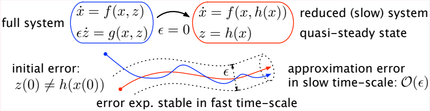

Singular Perturbation Analysis

Time-scale separation in power network model:

- **power network model:**

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

- singular perturbation parameter: $\epsilon = \frac{M_{\max}}{\pi f_0 D_{\min}}$
- reduced system for $\epsilon = 0$ is a **non-uniform Kuramoto model**:

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Tikhonov's Theorem:
Assume the non-uniform Kuramoto model synchronizes exponentially. Then $\forall\, (\theta(0), \dot{\theta}(0))$ there exists $\epsilon^* > 0$ such that $\forall\, \epsilon < \epsilon^*$ and $\forall\, t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

Singular Perturbation Analysis

Discussion of the assumption $\epsilon = \frac{M_{\max}}{\pi f_0 D_{\min}}$ sufficiently small:

1. physical interpretation: damping and sync on separate time-scales
2. classic assumption in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
3. physical reality: with generator internal control effects $\epsilon \in \mathcal{O}(0.1)$
4. simulation studies show accurate approximation even for large ϵ
5. non-uniform Kuramoto model corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ used successfully in academia and industry since 1978

Outline

Main Synchronization Result

Conditions on network parameters:

network connectivity > network's non-uniformity + network's losses,
and gap determines domain of attraction

1. **Non-Uniform Kuramoto Model:**

$\Rightarrow$ exponential synchronization: phase locking & frequency entrainment

$\Rightarrow$ for $\varphi_{ij} = 0$: explicit synchronization frequency & synchronization rates

$\Rightarrow$ for $\varphi_{ij} = 0$ & $\omega_i = \omega_j$: exponential phase synchronization

2. **Power Network Model:**

$\Rightarrow$ there exists ϵ sufficiently small such that for all $t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

$\Rightarrow$ for ϵ and network losses φ_{ij} sufficiently small, $\mathcal{O}(\epsilon)$ error converges

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Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: D_i, ω_i, P_{ij}, φ_{ij}
- **Directed coupling** between oscillator i and j:
  - coupling weights: $\frac{P_{ij}}{D_i} \neq \frac{P_{ij}}{D_j}$
  - coupling functions: $\sin(\theta_i - \theta_j + \varphi_{ij}) + \sin(\theta_j - \theta_i + \varphi_{ij}) \neq 0$
- **Phase shift** φ_{ij} induces lossless and lossy couling:
  $P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij}) = P_{ij}\cos(\varphi_{ij})\sin(\theta_i - \theta_j) + P_{ij}\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$
- **Synchronization analysis** in multiple steps:
  1. phase locking: $\theta_i(t) - \theta_j(t)$ becomes bounded
  2. frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
  3. phase synchronization: $\theta_i(t) - \theta_j(t) \to 0$

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j\neq i} \frac{P_{ij}}{D_i}\cos(\varphi_{ij})\sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$$

Condition (1) for synchronization:
Assume the graph induced by $P = P^T$ is complete and

$$\underbrace{n\frac{P_{\min}}{D_{\max}}\cos(\varphi_{\max})}_{\text{worst lossless coupling}} > \underbrace{\max_{\{i,j\}}\left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\right)}_{\text{worst non-uniformity}} + \underbrace{\max_i \sum\nolimits_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij})}_{\text{worst lossy coupling}}.$$

Gap determines the admissible initial lack of phase locking in a $\frac{\pi}{2}$ interval.

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j)$$

Condition (1) for synchronization:

$$K > \omega_{\max} - \omega_{\min}$$

Gap determines the admissible initial lack of phase locking in a $\frac{\pi}{2}$ interval.

Condition (1) strictly improves existing bounds on Kuramoto model:
[F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, J.L. van Hemmen et al. '93].

Necessary condition for sync of n oscillators: $K > \frac{n}{2(n-1)}(\omega_{\max} - \omega_{\min})$
[J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, N. Chopra et al. '09]

Theorem: Phase locking and frequency entrainment (1)

Non-uniform Kuramoto with complete $P = P^T$
Assume minimal coupling larger than a critical value, i.e.,

$$P_{\min} > P_{\text{critical}} := \frac{D_{\max}}{n\cos(\varphi_{\max})}\left(\max_{\{i,j\}}\left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\right) + \max_i \sum\nolimits_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\right)$$

Define $\delta = \frac{\pi}{2} - \arccos\left(\cos(\varphi_{\max})\frac{P_{\text{critical}}}{P_{\min}}\right)$ and set of locked phases

$$\Delta(\delta) := \{\theta \in \mathbb{T}^n | \max_{\{i,j\}} |\theta_i - \theta_j| \leq \delta\}$$

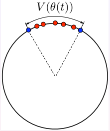
Then

1) **phase locking**: the set $\Delta(\delta)$ is positively invariant
2) **frequency entrainment:** $\forall\, \theta(0) \in \Delta(\delta)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$

Synchronization of Non-Uniform Kuramoto Oscillators

Main proof ideas:

1. **Phase locking** in $\Delta(\delta) \Leftrightarrow$ arc-length $V(\theta(t))$ is non-increasing



$$\Leftrightarrow \begin{cases} V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)| \mid i,j \in \{1,\dots,n\}\} \\ D^+ V(\theta(t)) \overset{!}{\leq} 0 \end{cases}$$

$\sim$ **contraction property** from consensus literature: [D. Bertsekas et al. '94, L. Moreau '04 & '05, Z. Lin et al. '08, ...]

2. **Frequency entrainment** in $\Delta(\delta) \Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\ddot{\theta}_i = -\sum\nolimits_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{P_{ij}}{D_i}\cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0$ for all $t \geq 0$

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Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \frac{P_{ij}}{D_i}\cos(\varphi_{ij})\sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i}\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$$

Condition (2) for synchronization:
Assume the graph induced by $P = P^T$ is connected with unweighted Laplacian L and weighted Laplacian $L(P_{ij}\cos(\varphi_{ij}))$ and

$$\underbrace{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))}_{\text{lossless connectivity}} > \underbrace{f(D_i)}_{\text{non-uniform } D_i\text{s}} \cdot \underbrace{(1/\cos(\varphi_{\max}))}_{\text{necessary phase locking}} \times$$
$$\Big(\underbrace{\Big\|\Big[\dots, \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \dots\Big]\Big\|_2}_{\text{non-uniformity}} + \underbrace{\sqrt{\lambda_{\max}(L)}\Big\|\Big[\dots, \sum\nolimits_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij}), \dots\Big]\Big\|_2}_{\text{lossy coupling}}\Big)$$

Gap determines the admissible initial lack of phase locking in a π interval.

Synchronization of Non-Uniform Kuramoto Oscillators

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j \neq i}\sin(\theta_i - \theta_j)$$

Condition (2) for synchronization:

$$K > \|[\dots, \omega_i - \omega_j, \dots]\|_2$$

Gap determines the admissible initial lack of phase locking in a π interval.

Condition (2) corresponds to the bound in [N. Chopra et al. '09].

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: Phase locking and frequency entrainment (2)

Assume graph induced by $P = P^T$ is connected with unweighted Laplacian L, incidence matrix H, and weighted Laplacian $L(P_{ij}\cos(\varphi_{ij}))$.
Assume algebraic connectivity is larger than a critical value, i.e.,

$$\lambda_2(L(P_{ij}\cos(\varphi_{ij}))) > \lambda_{\text{critical}} := \frac{\|HD^{-1}\omega\|_2 + \sqrt{\lambda_{\max}(L)}\,\|[\ldots, \sum_j \frac{P_{ij}}{D_i}\sin(\varphi_{ij}), \ldots]\|_2}{\cos(\varphi_{\max})(\kappa/n)\mu \min_{\{i,j\}}\{D_{\neq\{i,j\}}\}},$$

$$\text{where} \quad \kappa := \sum_{k=1}^{n} \frac{1}{D_{\neq k}}, \quad \mu := \sqrt{\min_{i\neq j}\{D_i D_j\}/\max_{i\neq j}\{D_i D_j\}}$$

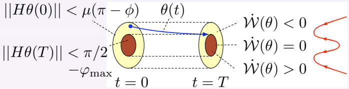
Define $\phi_{\min} \in (0, \frac{\pi}{2})$ by $\text{sinc}(\pi - \phi_{\min}) = (2/\pi)\lambda_{\text{critical}}/\lambda_2(L(P_{ij}\cos(\varphi_{ij})))$.

1) **phase locking:** $\forall\ \|H\theta(0)\|_2 \leq \mu(\pi - \phi_{\min})$, there is $T \geq 0$ such that $\|H\theta(t)\|_2 < \frac{\pi}{2} - \varphi_{\max}$ for all $t > T$
2) **frequency entrainment:** $\forall\ \|H\theta(0)\|_2 \leq \mu(\pi - \phi_{\min})$ the frequencies $\dot\theta_i(t)$ synchronize exponentially to some frequency $\dot\theta_\infty \in [\dot\theta_{\min}(0), \dot\theta_{\max}(0)]$

Synchronization of Non-Uniform Kuramoto Oscillators

Main proof ideas:

1. **Phase locking** in $\Delta(\delta)$ via ultimate boundedness arguments



$$\mathcal{W}(\theta) = \frac{1}{2}\sum_{\{i,j\}} \frac{1}{D_{\neq\{i,j\}}}|\theta_i - \theta_j|^2$$

2. **Frequency entrainment** for $t > T \Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\ddot\theta_i = -\sum_{j\neq i} a_{ij}(t)(\dot\theta_i - \dot\theta_j),$$

$$\text{where } a_{ij}(t) = \frac{P_{ij}}{D_i}\cos(\theta_i(t) - \theta_j(t) + \varphi_{ij}) > 0 \text{ for all } t > T$$

Outline

1. Introduction
   1. synchronization and transient stability
   2. power network model
   3. consensus and Kuramoto oscillators
2. Singular perturbation analysis
   (to relate power network and Kuramoto model)
3. Synchronization analysis (of non-uniform Kuramoto model)
   1. Main synchronization result
   2. Sufficient condition (based on weakest lossless coupling)
   3. Sufficient condition (based on lossless algebraic connectivity)
   4. **Further results**
4. Conclusions

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A refined result on frequency entrainment

Assume graph induced by P has globally reachable node and there exists $\delta \in (0, \frac{\pi}{2})$ such that the phases are locked in the set $\Delta(\delta)$

If $P = P^T$ & $\varphi_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$, then $\forall\ \theta(0) \in \Delta(\delta)$ the frequencies $\dot\theta_i(t)$ synchronize exp. to the weighted mean frequency

$$\Omega_c = \frac{1}{\sum_i D_i}\sum_i D_i\omega_i$$

and the exponential synchronization rate is no worse than

$$\lambda_{\text{fe}} = -\underbrace{\lambda_2(L(P_{ij}))}_{\text{connectivity}}\ \underbrace{\sin(\delta)}_{\Delta(\delta)}\ \underbrace{\cos(\angle(D\mathbf{1},\mathbf{1}))^2}_{\mathbf{1}\not\perp D\mathbf{1}} / \underbrace{D_{\max}}_{\text{slowest}}$$

Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A result on phase synchronization

Assume the graph induced by P has a globally reachable node, and $\varphi_{ij} = 0$ and $\frac{\omega_i}{D_i} = \frac{\omega_j}{D_j}$ for all $i, j \in \{1, \dots, n\}$. Let $\phi \in (0, \pi]$.

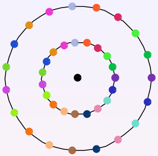
For the non-uniform Kuramoto model,

1) $\forall\, \theta(0) \in \{\theta \in \mathbb{T}^n : \max_{\{i,j\}} |\theta_i - \theta_j| < \pi - \phi\}$ the phases $\theta_i(t)$ synchronize exponentially; and
2) if $P = P^T$, $\forall\ \|H\theta(0)\|_2 \leq \mu(\pi - \phi)$ the phases $\theta_i(t)$ synchronize exponentially at a rate no worse than

$$\lambda_{\text{ps}} = -\underbrace{(\kappa/n)\min_{\{i,j\}}\{D_{\neq\{i,j\}}\}}_{\text{weigting of } D_i}\ \underbrace{\text{sinc}(\pi - \phi)}_{\theta(0)}\ \underbrace{\lambda_2(L(P_{ij}))}_{\text{connectivity}}$$

Result can be reduced to [A. Jadbabaie et al. '04].

Simulation Studies



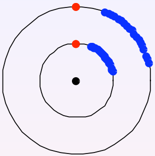
Simulation data:

- worst-case initial phase-differences: $\theta_i(0)$ in splay state
- $\epsilon = 0.12$s is small
- **strongly** non-uniform network

$\Rightarrow$ sufficient conditions for synchronization are **not satisfied**

Result: singular perturbation analysis is accurate ✓
both models synchronize ✓

Simulation Studies



Simulation data:

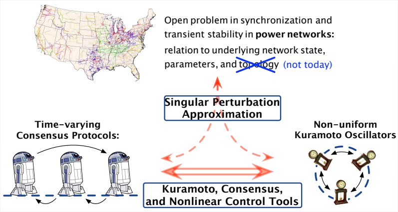
- initial phases mostly clustered besides red phasor
- $\epsilon = 0.6$s is **large**
- non-uniform network

$\Rightarrow$ sufficient conditions for synchronization are **satisfied**

Result: singular perturbation analysis is accurate ✓
both models synchronize ✓

Conclusions

Summary:



Future Work:

- relation to network topology, clustering and scalability
- synchronization in optimal power flow problems